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Properties of period integrals of complex algebraic curves

José M. Isidro *

Dipartimento di Fisica "G. Galilei", Via Marzolo 8, 15131 Padova, Italy

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Abstract

The period integrals of non-singular complex algebraic curves in $\mathbb{C}^2$ are shown to satisfy a set of polynomial relations that can be used to formulate the corresponding the Picard–Fuchs equations. Their derivation employs elementary mathematical techniques that have also found application in the context of Seiberg–Witten theories.

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1. Introduction

Effective $N = 2$ supersymmetric Yang–Mills theories in four dimensions have been the subject of great attention recently [1]. These models, also called Seiberg–Witten (SW) theories, exhibit a wide variety of interesting physical and mathematical properties. In many SW theories the moduli space of physical vacua in the Coulomb branch coincides with the moduli space of a certain class of Riemann surfaces Σ_g. In their simplest examples, these Riemann surfaces are hyperelliptic [2]. Such is the situation when one considers a simple, classical gauge group, with or without matter hypermultiplets in the defining representation. Some configurations with classical gauge groups given by the product of several simple factors, or with matter hypermultiplets in representations other than the fundamental, are known to be described by non-hyperelliptic Riemann surfaces, and have been studied recently in the context of geometric engineering [3,4] and M-theory [5–7]. In other cases, the moduli space of vacua does not correspond to a family of Riemann surfaces, but to some different geometric manifolds [3]. In this paper we will restrict

* E-mail: isidro@pd.infn.it

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ourselves to those cases that correspond to Riemann surfaces, although not necessarily hyperelliptic.

An essential ingredient in these constructions is the SW differential and its period integrals along (a subset of) the homology cycles of Σ_g [1,2]. In principle, a knowledge of these period integrals amounts to the complete solution of the effective theory in question, as given by the full quantum prepotential $\mathcal{F}$. Different approaches have been undertaken in order to compute $\mathcal{F}$. One of them makes use of a set of partial, second-order differential equations (with respect to the moduli) satisfied by the periods of the SW differential [8,9]. The differential equations just mentioned are known as Picard–Fuchs (PF) equations. They can be traced back to a first-order system of differential equations satisfied by the holomorphic differentials on the Riemann surface, that also goes by the same name of PF equations. The latter have been known for some time in the mathematical literature [10,11] as the equations governing the variation of a Hodge structure on the manifold. It is therefore of some interest to study the properties of such period integrals without regard to their particular physical origin. This may help to highlight their wide range of applicability. For example, a number of properties of hyperelliptic period integrals, related to some of those reported here, were observed in [12] in the (apparently unrelated) context of conformal field theory. We believe this may be an interesting offshoot of SW models that connects with fields such as classical Riemann surface theory [13], string theory on general algebraic curves [14], special geometry, mirror symmetry and Calabi–Yau manifolds (see [15] and references therein).

In previous work [16], a technique has been developed in order to give a systematic derivation of the PF equations corresponding to those SW models that can be described by hyperelliptic Riemann surfaces. This method is based on a set of algebraic relations satisfied by the periods of the SW differential on the corresponding surface. In this paper we formalise the method outlined in [16] and extend it to an arbitrary (non-singular) Riemann surface, be it hyperelliptic or not. We provide some technical proofs omitted in [16], while recasting our presentation in a way that is independent of its possible connections with SW theory, or any other applications. We begin in Section 2 with a detailed treatment of the hyperelliptic case, where an expression is given for an arbitrary modular derivative of the period of an arbitrary holomorphic differential on Σ_g. Section 3 deals with the same problem in the non-hyperelliptic case; although the technical details are more involved, the method is the same as in the hyperelliptic case. Some concluding remarks are made in Section 4.

Throughout our analysis, u_i will denote an arbitrary modulus (or set of moduli) whose significance may vary. Thus, e.g., in the context of SW models, specifying a value for u_i is equivalent to determining a physical vacuum state of a certain effective $N = 2$ supersymmetric Yang–Mills theory. Alternatively, our techniques would apply just as well to the mathematical problem of varying a conformal structure on Σ_g.

2. Hyperelliptic Riemann surfaces

Let $p(x)$ be the complex polynomial

$$p(x) = \prod_{l=1}^{2g+1}(x - e_l) = \sum_{j=0}^{2g+1} s_j x^{2g+1-j}. \tag{2.1}$$

where $g \geq 1$. The coefficients s_j are the symmetric polynomials in the roots e_l. The discriminant Δ of $p(x)$ is the function of the roots of $p(x)$ given by

$$\Delta = \prod_{l<n}^{2g+1} (e_l - e_n)^2. \tag{2.2}$$

Assume $\Delta \neq 0$, i.e., $e_l \neq e_n$ if $l \neq n$. Then the equation

$$y^2 = p(x) \tag{2.3}$$

defines a family of non-singular hyperelliptic Riemann surfaces of genus g, Σ_g. Each one of them is a twofold covering of the Riemann sphere S^2 branched over the roots e_l, plus over the point at infinity. Choices of the roots e_l (or, alternatively, of the s_j) such that the discriminant vanishes will produce singular surfaces, in that some of the 1-cycles of the homology $H_1(\Sigma_g)$ will collapse to a point. We will assume $\Delta(s_j) \neq 0$ in all what follows.

The differential 1-forms on Σ_g

$$\omega_n = x^n \frac{\mathrm{d}x}{y}, \qquad n = 0, 1, 2, \ldots \tag{2.4}$$

are holomorphic for $0 \leq n \leq g - 1$, while they are meromorphic with vanishing residues for all $n \geq g$. Let $\gamma \in H_1(\Sigma_g)$ be an arbitrary 1-cycle. The period integrals

$$\Omega_n(\gamma) = \int_\gamma \omega_n, \tag{2.5}$$

and the differential equations they satisfy, will be our focus of attention. In order to derive the latter, a generalisation of equations (2.4) and (2.5) is needed. Let $\mathbb{Z}^+$ denote the non-negative integers and $\frac{1}{2}\mathbb{Z}^-$ the negative half-integers (i.e., the set $-1/2, -3/2, \ldots$).

Consider $\mu \in \frac{1}{2}\mathbb{Z}^-$ and $n \in \mathbb{Z}^+$, and let a given 1-cycle $\gamma \in H_1(\Sigma_g)$ be fixed. Define the μ-period of x^n along γ, denoted by $\Omega_n^{(\mu)}(\gamma)$, as

$$\Omega_n^{(\mu)}(\gamma) := (-1)^{\mu+1} \Gamma(\mu + 1) \int_\gamma \frac{x^n}{p^{\mu+1}(x)} \mathrm{d}x. \tag{2.6}$$

In Eq. (2.6), Γ stands for Euler's gamma function. The overall normalisation factor $(-1)^{\mu+1}$ $\Gamma(\mu + 1)$ has been included for convenience. The usual period matrix of Σ_g (in the sense of Eq. (2.5)) is obtained when we set $\mu = -1/2$ and let γ run over a basis of $H_1(\Sigma_g)$ [13], but for the moment, μ will remain an arbitrary negative half-integer. One can easily prove that $\Omega_n^{(\mu)}(\gamma)$ is well defined as a function of the homology class of γ.

Let $\gamma \in H_1(\Sigma_g)$ be given. Then the periods $\Omega_n^{(\mu)}(\gamma)$ satisfy the following recursion relations:

$$\Omega_n^{(\mu)}(\gamma) = \frac{1}{n + 1 - (1 + \mu)(2g + 1)} \sum_{j=0}^{2g-1} j s_j \Omega_{n+2g-1-j}^{(\mu+1)}(\gamma), \tag{2.7}$$

and

$$\Omega_n^{(\mu+1)}(\gamma) = \frac{1}{n-2g-(1+\mu)(2g+1)} \times \sum_{j=1}^{2g+1} [(1+\mu)(2g+1-j) - (n-2g)] s_j \Omega_{n-j}^{(\mu+1)}(\gamma). \tag{2.8}$$

In order to prove them, we observe that from Eq. (2.1) one can write

$$x^{2g} = \frac{1}{2g+1}\left[\frac{\partial p(x)}{\partial x} - \sum_{j=1}^{2g+1}(2g+1-j)s_j x^{2g-j}\right]. \tag{2.9}$$

Multiply Eq. (2.9) by $x^n/p^{\mu+2}(x)$ and take its line integral along γ. Upon integration by parts and dropping a total derivative, one obtains

$$\Omega_{n+2g}^{(\mu+1)}(\gamma) = -\frac{n}{2g+1}\Omega_{n-1}^{(\mu)}(\gamma) - \frac{1}{2g+1} \times \sum_{j=1}^{2g+1}(2g+1-j)s_j \Omega_{n+2g-j}^{(\mu+1)}(\gamma). \tag{2.10}$$

Solve Eq. (2.10) for $\Omega_{n-1}^{(\mu)}(\gamma)$ and shift $n \to n+1$ to obtain

$$\Omega_n^{(\mu)}(\gamma) = \frac{-1}{n+1}\left[(2g+1)\Omega_{n+2g+1}^{(\mu+1)}(\gamma) + \sum_{j=1}^{2g+1}(2g+1-j)s_j \Omega_{n+2g+1-j}^{(\mu+1)}(\gamma)\right]. \tag{2.11}$$

Next consider

$$\begin{aligned} -(1+\mu)\Omega_n^{(\mu)}(\gamma) &= (-1)^{\mu+2}\Gamma(\mu+2)\int_\gamma \frac{x^n\, p(x)}{p^{\mu+2}(x)}\,\mathrm{d}x \\ &= \Omega_{n+2g+1}^{(\mu+1)}(\gamma) + \sum_{j=1}^{2g+1} s_j \Omega_{n+2g+1-j}^{(\mu+1)}(\gamma), \end{aligned} \tag{2.12}$$

from where

$$\Omega_{n+2g+1}^{(\mu-1)}(\gamma) = -(1+\mu)\Omega_n^{(\mu)}(\gamma) - \sum_{j=1}^{2g+1} s_j \Omega_{n+2g+1-j}^{(\mu-1)}(\gamma). \tag{2.13}$$

Substitution of Eq. (2.13) into (2.11) produces the recursion relation (2.7). Finally, the recursion (2.8) is obtained by substitution of Eq. (2.7) into (2.13) after shifting $n \to n-2g-1$.

Two observations are worth making. First, the above holds for any $\mu \in \frac{1}{2}\mathbb{Z}^-$ and any $n \in \mathbb{Z}^+$. For such values of μ and n, the denominators of (2.7) and (2.8) never vanish.

Second, it ceases to hold on the zero locus of the discriminant, i.e., when $\Delta(s_j) = 0$. This is a consequence of the following decomposition [17] for the discriminant $\Delta(s_j)$ of the curve:

$$\Delta(s_j) = a(x)p(x) + b(x)\frac{\partial p(x)}{\partial x}, \tag{2.14}$$

where $a(x)$ and $b(x)$ are certain polynomials in x. We have $\Delta(s_j) = 0$ if, and only if, both $p(x)$ and $\partial p(x)/\partial x$ vanish simultaneously. In this case the derivation of the recursion relations (2.7) and (2.8) breaks down.

We define the *basic range* R to be the set of all $n \in \mathbb{Z}^-$ such that $0 \leq n \leq 2g - 1$. Let $Z[s_j]$ be the ring of polynomials in the s_j with complex coefficients.

The set of all periods $\Omega_n^{(\mu+1)}(\gamma)$ as n runs over $\mathbb{Z}^+$, when both μ and γ are kept fixed, defines a module over $Z[s_j]$; let it be denoted by $\mathcal{M}^{(\mu+1)}(\gamma)$. Then Eqs. (2.7) and (2.8) prove that $\dim \mathcal{M}^{(\mu+1)}(\gamma) = 2g$. This follows from the observation that Eq. (2.8) will express any $\Omega_n^{(\mu+1)}(\gamma)$, where $n \geq 2g$, as a certain linear combination of $\Omega_m^{(\mu+1)}(\gamma)$, with $m < n$, and with coefficients that will be certain homogeneous polynomials in the s_j. Repeated application of this recursion will eventually allow to reduce all terms in that linear combination to a sum over the periods of the basic range R. The $\Omega_n^{(\mu+1)}(\gamma)$, where $n \in R$, will be called *basic periods* of $\mathcal{M}^{(\mu+1)}(\gamma)$.

The two modules $\mathcal{M}^{(\mu)}(\gamma)$ and $\mathcal{M}^{(\mu+1)}(\gamma)$ are clearly isomorphic. We can make this isomorphism more explicit as follows. Let $\Omega_n^{(\mu)}(\gamma)$ and $\Omega_m^{(\mu+1)}(\gamma)$, where both n and m run over R, be bases of $\mathcal{M}^{(\mu)}(\gamma)$ and $\mathcal{M}^{(\mu+1)}(\gamma)$, respectively. Arrange them as column vectors $\Omega^{(\mu)} = (\Omega_0^{(\mu)}, \Omega_1^{(\mu)}, \ldots, \Omega_{2g-1}^{(\mu)})^t$ and $\Omega^{(\mu+1)} = (\Omega_0^{(\mu+1)}, \Omega_1^{(\mu-1)}, \ldots, \Omega_{2g-1}^{(\mu-1)})^t$. For notational simplicity we have suppressed the dependence on γ. Now, for every $\mu \in \frac{1}{2}\mathbb{Z}^-$ there exists a unique matrix $M^{(\mu)}$ such that

$$\Omega^{(\mu)} = M^{(\mu)}\Omega^{(\mu+1)}. \tag{2.15}$$

The entries of $M^{(\mu)}$ are certain homogeneous polynomials in the s_j. $M^{(\mu)}$ is non-singular when $\Delta(s_j) \neq 0$.

The statement (2.15) can be proved as follows. The existence and uniqueness of $M^{(\mu)}$ are a consequence of Eqs. (2.7) and (2.8). Taking $n \in R$, the period $\Omega_n^{(\mu)}$ can be expressed as a linear combination of some $\Omega_m^{(\mu+1)}$, as per Eq. (2.7), with certain homogeneous polynomials in the s_j as coefficients, but with m not necessarily in R. Then use Eq. (2.8) as many times as necessary, in order to pull the subindex m of $\Omega_m^{(\mu+1)}$ back into R. The coefficients in this expansion are certain homogeneous polynomials in the s_j, explicitly computable from Eqs. (2.7) and (2.8). They define the rows of the matrix $M^{(\mu)}$ as n runs over R. The invertibility of $M^{(\mu)}$ when $\Delta(s_j) \neq 0$ follows from the fact that $\Omega_n^{(\mu)}$ is a basis of $\mathcal{M}^{(\mu)}$ when $n \in R$ and $\Omega_m^{(\mu+1)}$ is a basis of $\mathcal{M}^{(\mu+1)}$ when $m \in R$.

As a function of the s_j, we have proved that $\det M^{(\mu)}$ can only vanish when $\Delta(s_j) = 0$. The entries of $M^{(\mu)}$ are polynomials in the s_j, hence $\det M^{(\mu)}$ will also be a polynomial in the s_j. Decompose $\Delta(s_j)$ into irreducible factors. Up to an overall complex constant, $\det M^{(\mu)}$ must therefore decompose as a product of exactly those same factors present in the decomposition of $\Delta(s_j)$, possibly with different multiplicities (eventually with zero multiplicity, i.e., $\Delta(s_j)$ might have more zeroes than $\det M^{(\mu)}$).

Next we consider derivatives of periods. As a consequence of Eqs. (2.6) and (2.1) we have, after commuting the derivative $\partial/\partial s_j$ past the integral sign,

$$\frac{\partial \Omega_n^{(\mu)}}{\partial s_j} = \Omega_{n+2g+1-j}^{(\mu+1)}. \tag{2.16}$$

Now take $n \in R$ in Eq. (2.16), and use the recursion relation (2.8) as many times as necessary in order to pull the subindex $n + 2g + 1 - j$ back into R. As n runs over R, the right-hand side of Eq. (2.16) defines the rows of a $(2g \times 2g)$-dimensional matrix, $D_j^{(\mu)}$. In matrix form, Eq. (2.16) reads

$$\frac{\partial \Omega^{(\mu)}}{\partial s_j} = D_j^{(\mu)}\, \Omega^{(\mu-1)}. \tag{2.17}$$

Since Eq. (2.15) can be inverted when $\Delta(s_j) \neq 0$, we have

$$\frac{\partial \Omega^{(\mu)}}{\partial s_j} = D_j^{(\mu)}\, (M^{(\mu)})^{-1}\, \Omega^{(\mu)} = S_j^{(\mu)}\, \Omega^{(\mu)}. \tag{2.18}$$

where we have defined $S_j^{(\mu)} = D_j^{(\mu)}\, (M^{(\mu)})^{-1}$. We finally set $\mu = -1/2$ in order to obtain the PF equations of the hyperelliptic Riemann surface (2.3):

$$\frac{\partial}{\partial s_j} \Omega^{(-1/2)} = S_j^{(-1/2)}\, \Omega^{(-1/2)}. \tag{2.19}$$

They express the derivatives of the basic periods with respect to the s_j, as certain linear combinations of the same basic periods. The $4g^2$ entries of the matrix $S_j^{(-1/2)}$ are certain rational functions of the s_j, explicitly computable using the recursion relations above. Finally, from a knowledge of the coefficients s_j as functions of the moduli u_i, application of the chain rule and Eq. (2.19) produces the desired PF equations $\partial\Omega/\partial u_i$.

To close this section, we would like to observe that results similar to those established here have been reported in [12], although using different techniques. An important difference in our approach lies in the fact that we have chosen to place one branching point at $x = \infty$. As the multiplicity of the branching is always maximal (and equal to 2) when the surface is hyperelliptic, placing one branching point at infinity is no stringent condition at all. One can always Moebius-transform the sphere S^2 in order to ensure that infinity becomes a branching point with maximal multiplicity. In turn, this guarantees that the family of differentials of Eq. (2.4) has vanishing residues everywhere on Σ_g. The issue of residues at infinity and the branching properties at this point will be relevant in Section 3, so let us briefly review our argument.

On the points of Σ_g that lie at finite distance, the differential ω_n can only have zeroes. Whatever poles ω_n may have, if any, will be at infinity. This can be seen as follows. At finite points, the poles of ω_n will be at the zeroes of the denominator y, that is, at the branching points. However, at the latter one can always use Eq. (2.3) to reexpress ω_n as

$$\omega_n = 2x^n \frac{\mathrm{d}y}{p'(x)}. \tag{2.20}$$

with $p'(x) \neq 0$ because $\Delta \neq 0$. This proves the analyticity of ω_n away from infinity. For $n \geq g$, ω_n does have poles at infinity, but with zero residue. This follows from the fact [13] that, for all n, $\sum_p \mathrm{Res}_p\,(\omega_n) = 0$, where the sum extends over all points $p \in \Sigma_g$. On the other hand, we have just seen that $\mathrm{Res}_p\,(\omega_n)$ can be non-zero only at $p = \infty$. Since the latter has been chosen to be a branching point, we conclude that $\mathrm{Res}_\infty\,(\omega_n) = 0$ for all n.

3. Non-hyperelliptic Riemann surfaces

Let us now consider the complex algebraic curve defined in $\mathbb{C}^2$ by the zero locus of the irreducible polynomial

$$F(x, y) = \sum_{j=0}^{n} p_j(x) y^{n-j} = p_0(x) y^n + p_1(x) y^{n-1} + \cdots + p_n(x), \tag{3.1}$$

where $n > 2$, and the polynomial $p_j(x)$ is assumed to be of degree j at most:

$$p_j(x) = \sum_{m=0}^{j} s_{jm}\, x^{j-m}. \tag{3.2}$$

Without loss of generality we can set $p_0(x) = 1$. As in Section 2, we will assume the curve (3.1) to be non-singular, i.e., $F(x, y)$, $F_x = \partial F/\partial x$ and $F_y = \partial F/\partial y$ cannot all vanish simultaneously.

We have already seen in our treatment of the hyperelliptic case that the behaviour of the curve at infinity controls the analytic properties of the family of differentials under consideration. The same will be true in the non-hyperelliptic case. However, on the non-hyperelliptic Riemann surface defined by $F(x, y) = 0$, the multiplicity at a given branching point can take any value from 2 to n. There may, or there may not, be branching points with maximal multiplicity n. A Moebius transformation on S^2 can bring $x = \infty$ to lie below any given branching point, but this still leaves the multiplicity at infinity free to vary from 2 to n. As will be justified presently, now we will find it convenient to pick infinity *not* a branching point. A sufficient condition that ensures this property is the following. Consider the polynomial $Q(t)$ defined by

$$Q(t) = \sum_{j=0}^{n} s_{j0}\, t^{n-j} = t^n + s_{10} t^{n-1} + \cdots + s_{n0}, \tag{3.3}$$

and assume it has n distinct roots in t. One can then prove [17] that Σ_g has n distinct points above $x = \infty \in S^2$, and that its genus is

$$g = \tfrac{1}{2}(n-1)(n-2). \tag{3.4}$$

Obviously not every value of the genus g can be written as above. One must allow for singularities if one wants (3.1) to describe an arbitrary value of g [17]. However, under the assumption of non-singularity we are making one can prove that, after suitable algebraic transformations, every non-singular algebraic curve in $\mathbb{C}^2$ can be cast into the form considered above [17].

Let us now consider the family of differentials on Σ_g given by

$$\omega_{jm} = y^j x^m \frac{\mathrm{d}x}{F_y}, \quad 0 \le m, \ 0 \le j. \tag{3.5}$$

In order to study the analytic properties of ω_{jm} we compute its divisor $[\omega_{jm}]$.[1] At finite points ω_{jm} cannot have poles. We prove this as follows. From $\mathrm{d}F(x, y) = 0$, the identity

$$\frac{\mathrm{d}y}{F_x} = -\frac{\mathrm{d}x}{F_y} \tag{3.6}$$

and the assumption of non-singularity give an alternative expression for ω_{jm} at the branching points, i.e., at the simultaneous solutions of $F(x, y) = 0$ and $F_y(x, y) = 0$. So, at finite points, ω_{jm}, can only have zeroes. Points where $x = 0$ are zeroes with order m, while points where $y = 0$ are zeroes with order j if $p_n(x) = 0$. The behaviour at infinity can be easily determined from our assumption that it is not a branching point. Then F_y has a pole of order $n - 1$ at infinity. Altogether, if the roots of $p_n(x)$ are denoted by q_s, the divisor $[\omega_{jm}]$ is

$$[\omega_{jm}] = j \sum_{s=1}^{n} q_s + m \sum_{l=1}^{n} 0_l + [(n-3) - (j+m)] \sum_{l=1}^{n} \infty_l, \tag{3.7}$$

where 0_l and ∞_l denote the points on the lth sheet of Σ_g lying above $x = 0$ and $x = \infty$ on S^2, respectively. We see that ω_{jm} has poles at infinity if $j + m > n - 3$, while it is holomorphic if $j + m \le n - 3$. The number of the latter equals $(n-1)(n-2)/2$, so according to Eq. (3.4) they provide a basis of holomorphic 1-forms.

Following Eq. (2.6), we define

$$\Omega_{jm}^{(\mu)}(\gamma) := (-1)^{\mu+1} \Gamma(\mu + 1) \int_{\gamma} \frac{y^j x^m}{(F_y)^{\mu+1}} \, \mathrm{d}x. \tag{3.8}$$

Setting $\mu = 0$ we recover the usual period matrix of Σ_g. However, as in Section 2, we will formally work with an arbitrary $\mu \in \mathbb{Z}$, which we will only set to zero at the very end.

We intend to derive recursions that relate periods $\Omega_{jm}^{(\mu)}(\gamma)$ with different values of the indices j, m and μ. The latter are well defined as functions of the homology class of γ only if the integrand is residue-free everywhere on Σ_g. We saw in the hyperelliptic case that a way of avoiding the residues at infinity was to choose it as a branching point. In the non-hyperelliptic case treated in this section, choosing infinity *not* to be a branching point causes the differentials ω_{jm} to develop poles at infinity when $j + m > n - 3$, possibly with non-zero residues. One can also expect residues at infinity when $\mu \ne 0$ during intermediate steps of the derivation.

Let us for the moment assume that infinity *were* a branching point of maximal multiplicity n. Then from $\sum_p \mathrm{Res}_p(\omega_{jm}) = 0$ we would conclude $\mathrm{Res}_\infty(\omega_{jm}) = 0$ for all ω_{jm}. This

[1] In our conventions, zeroes (poles) carry positive (negative) coefficients.

would provide us with a residue-free family of differentials to work with. However, in this case F_y would develop a zero of order $n-1$ at infinity, thus invalidating the divisor in Eq. (3.7). In fact, computing the divisor $[\omega_{jm}]$ as above shows that the differentials with $0 \leq j, 0 \leq m$ and $j+m \leq n-3$ would no longer be holomorphic, if infinity were to be a branching point of maximal multiplicity n. Similar difficulties can be expected if infinity is chosen to be a branching point with multiplicity smaller than n.

We see that there is an incompatibility between the ω_{jm} of Eq. (3.5) as a convenient basis of differentials to work with, on the one hand, and the requirement of having vanishing residues everywhere on Σ_g, on the other. The hyperelliptic case was an exception, in that a particular choice of branching points avoided this incompatibility.

To the effect of explicitly exhibiting recursion relations between periods, we will find it convenient to work with the differentials of Eq. (3.5). Next, assuming the polynomial $Q(t)$ in Eq. (3.3) has n distinct roots, we conclude that infinity is *not* a branching point, and a useful basis of holomorphic differentials is obtained when $0 \leq j, 0 \leq m$ and $j+m \leq n-3$. One further simplifying assumption we will make is that

$$p_n(x) = 1, \quad \text{i.e.,} \quad s_{nm} = \delta_{nm}. \tag{3.9}$$

Once the recursion relations have been established, the passage to a residue-free basis can be accomplished as in [12]. One considers a family of *normal differentials* ω_∞^{rs} [13] having residue $+1$ at point ∞_r on the rth sheet of Σ_g, and residue -1 at point ∞_s on the sth sheet, while being analytic elsewhere. With the help of these normal differentials, one can remove all residues at infinity by simply subtracting appropriate linear combinations of the ω_∞^{rs} from ω_{jm}.

Now let u_i be an arbitrary modulus. We would like to compute modular derivatives of the periods in Eq. (3.8). Reasoning as in Section 2 one finds

$$\frac{\partial}{\partial u_i}\Omega_{kl}^{(\mu)} = \sum_{j=0}^{n}\sum_{m=0}^{j}\left(n-j-\frac{k}{\mu+1}\right)\frac{\partial s_{jm}}{\partial u_i}\Omega_{k+n-j-1,l+j-m}^{(\mu+1)}. \tag{3.10}$$

One can again establish a linear relationship between the $\Omega^{(\mu)}$ and the $\Omega^{(\mu+1)}$, analogous to that of Eq. (2.15), with coefficients that are certain homogeneous polynomials in the s_{jm}. First we use Eqs. (3.1) and (3.2) to arrive at

$$\begin{aligned}\Omega_{kl}^{(\mu)} &= (-1)^{\mu+1}\Gamma(\mu+1)\int\frac{y^k x^l}{(F_y)^{\mu+2}}F_y\,\mathrm{d}x \\ &= \frac{-1}{\mu+1}\sum_{j=0}^{n}\sum_{m=0}^{j}(n-j)s_{jm}\Omega_{k+n-j-1,l+j-m}^{(\mu-1)}.\end{aligned} \tag{3.11}$$

The right-hand side of Eq. (3.11) just fails to define a matrix $M^{(\mu)}$ in the sense of Section 2. High values of the subindices will first have to be reduced into an irreducible set, i.e., into a basic range R, if Eq. (3.11) is to define the $M^{(\mu)}$ matrix properly. This can be done by means of two recursion relations (analogous to that of Eq. (2.8)) that we now derive.

From Eqs. (3.1) and (3.2) we can solve for the highest power of y to obtain

$$y^n = -\sum_{j=1}^{n}\sum_{m=0}^{j} s_{jm}\, x^{j-m}\, y^{n-j}. \tag{3.12}$$

Substituting Eq. (3.12) into the period $\Omega^{(\mu)}_{k+n,l}$ we find

$$\Omega^{(\mu)}_{k+n,l} = -\sum_{j=1}^{n}\sum_{m=0}^{j} s_{jm}\, \Omega^{(\mu)}_{k-n+j,l+j-m}. \tag{3.13}$$

The right-hand side of Eq. (3.13) reduces the first index with respect to the left-hand side. However, this is done at the cost of increasing the second index. We therefore need an independent recursion that will reduce the latter.

In order to derive it let us recall that $s_{n,0} = 0$ by Eq. (3.9). Then $Q(t)$ in Eq. (3.3) has $t = 0$ as a simple root if, and only if, $s_{n-1,0} \neq 0$. This allows us to identify the highest power of x in $F_{xy} = \partial^2 F/\partial x \partial y$ as the term $j = n-1$, $m = 0$ in the following sum:

$$\begin{aligned} F_{xy} &= \sum_{j=0}^{n}\sum_{m=0}^{j} (n-j)\,(j-m)\, s_{jm}\, x^{j-m-1}\, y^{n-j-1} \\ &= (n-1) s_{n-1,0} x^{n-2} + \sum_{m=1}^{n-1} (n-m-1)\, s_{n-1,m}\, x^{n-m-2} \\ &\quad + \sum_{j=0}^{n-2}\sum_{m=0}^{j} (n-j)\,(j-m)\, s_{jm}\, x^{j-m-1}\, y^{n-j-1}. \end{aligned} \tag{3.14}$$

From here we can write

$$\begin{aligned} &(n-1)\, s_{n-1,0}\, \Omega^{(\mu+1)}_{k,l+n-2} \\ &\quad = (-1)^{\mu+2}\Gamma(\mu+2) \\ &\quad \times \int \mathrm{d}x\, \frac{y^k\, x^l}{(F_y)^{\mu+2}} \left[F_{xy} - \sum_{m=1}^{n-1} (n-m-1)\, s_{n-1,m}\, x^{n-m-2} \right. \\ &\quad \left. - \sum_{j=0}^{n-2}\sum_{m=0}^{j} (n-j)\,(j-m)\, s_{jm}\, x^{j-m-1}\, y^{n-j-1} \right]. \end{aligned} \tag{3.15}$$

Let us analyse the term containing F_{xy} in Eq. (3.15). We have

$$\begin{aligned} &(-1)^{\mu+2}\Gamma(\mu+2) \int \frac{y^k\, x^l}{(F_y)^{\mu+2}}\, \frac{\partial F_y}{\partial x}\, \mathrm{d}x \\ &\quad = (-1)^{\mu+1}\Gamma(\mu+1) \int \frac{\partial}{\partial x}\left[\frac{1}{(F_y)^{\mu+1}}\right] x^l\, y^k\, \mathrm{d}x \\ &\quad = -l\, \Omega^{(\mu)}_{k,l-1} - k(-1)^{\mu+1}\Gamma(\mu+1) \int \frac{y^{k-1}\, x^l}{(F_y)^{\mu+1}}\, \frac{\mathrm{d}y}{\mathrm{d}x}\, \mathrm{d}x. \end{aligned} \tag{3.16}$$

where an integration by parts has been perfomed, and a total derivative dropped. From $\mathrm{d}F = 0$ we can express $\mathrm{d}y/\mathrm{d}x$ as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y} = -\frac{1}{F_y}\sum_{j=0}^{n}\sum_{m=0}^{j}(j-m)\, s_{jm}\, y^{n-j}\, x^{j-m-1}. \tag{3.17}$$

Substituting Eq. (3.17) into (3.16) one finds

$$(-1)^{\mu+2}\Gamma(\mu+2)\int \frac{y^k x^l}{(F_y)^{\mu+2}}\frac{\partial F_y}{\partial x}\,\mathrm{d}x$$
$$= -l\,\Omega^{(\mu)}_{k,l-1} - \frac{k}{\mu+1}\sum_{j=0}^{n}\sum_{m=0}^{j}(j-m)\, s_{jm}\, \Omega^{(\mu+1)}_{k+n-j-1,l+j-m-1}. \tag{3.18}$$

We finally use Eq. (3.18) in (3.15), apply Eq. (3.11) to express $\Omega^{(\mu)}_{k,l-1}$ in terms of periods $\Omega^{(\mu+1)}$, and pull all terms containing $\Omega^{(\mu+1)}_{k,l+n-2}$ to the left-hand side. After some lengthy but straightforward rearrangements one arrives at the following relation:

$$\left[\left(1+\frac{k}{\mu+1}\right)(n-1) - \frac{l}{\mu+1}\right] s_{n-1,0}\,\Omega^{(\mu+1)}_{k,l+n-2}$$
$$= \sum_{m=1}^{n-1}\left[\frac{l}{\mu+1} - \left(1+\frac{k}{\mu+1}\right)(n-m-1)\right] s_{n-1,m}\,\Omega^{(\mu+1)}_{k,l+n-m-2}$$
$$+\sum_{j=0}^{n-2}\sum_{m=0}^{j}\left\{\frac{1}{\mu+1}[l(n-j)+k(m-j)] + (j-n)(j-m)\right\}$$
$$\times\, s_{jm}\,\Omega^{(\mu+1)}_{k+n-j-1,l+j-m-1}. \tag{3.19}$$

We observe that all terms on the right-hand side of Eq. (3.19) contain a value of the second index smaller than $l+n-2$, which is its value on the left-hand side.

Once the above expressions have been obtained, the derivation of the PF equations goes through as in the hyperelliptic case after setting $\mu = 0$. The details will be omitted, but let us make some final observations.

The assumption that infinity is not a branching point is in fact no stringent condition at all. One can always Moebius-transform the sphere S^2 in order to have $x = \infty \in S^2$ *not* lying below any one branching point. Here we have chosen to ensure that this property holds by imposing certain algebraic conditions on the coefficients s_{jm} of the curve. The latter are sufficient, but not necessary conditions, that also turned out to be convenient when identifying the highest non-vanishing power of x in F_{xy} in Eq. (3.14). Had we made a different choice for the coefficients s_{jm}, compatibly with the property of infinity not a branching point, one could apply arguments similar to those given above in order to obtain a second recursion, like that in (3.19).

The recursions (3.13) and (3.19) will certainly increase one index while reducing the other. However, the number of iterations needed to reduce all periods to those within a basic range R is necessarily finite. In our case this follows from the observation that $F(x, y)$

is of degree n in y, while it is of degree $n - 1$ (at most) in x. As Eqs. (3.13) and (3.19) are certain polynomial relations based on $F(x, y)$ and F_{xy}, the difference in their degrees ensures that the rates of growth in x and y can never compensate each other. Therefore, a repeated application of these recursions will eventually reduce all indices into a set of irreducible values, i.e., into a basic range R. On more general grounds [13] one can argue that the space of periods (of differentials of the second kind) on a non-degenerate surface Σ_g is $2g$-dimensional. From this point of view, the recursion relations derived here exhibit this property very explicitly.

It is also possible to argue that the matrix $M^{(\mu)}$ defined by Eq. (3.11) (after reducing all indices into R) will be invertible away from the singularities of the curve. This follows from the same reasons as in the hyperelliptic case.

One could wonder if the two recursions (3.13) and (3.19) are at all independent, since x and y are "constrained" by $F(x, y) = 0$. Let us argue to the effect that they are indeed so. In the "constraint" $F(x, y) = 0$, decreasing powers of y go with increasing powers of x. This reflects itself in the fact that both relations (3.13) and (3.19) cause one index to increase while reducing the other one. Eq. (3.13) made a straight use of $F(x, y) = 0$ by simply solving for the highest power of y. Meanwhile, the recursion (3.19) first inserted F_{xy} into the period integral, then integrated by parts under the dx sign. A total derivative was dropped during the process. None of these operations could have been accomplished by a simple use of $F(x, y) = 0$. Incidentally, this mechanism also explains the choice of F_{xy} in order to derive the second recursion, Eq. (3.19). The x-derivative is accounted for in the partial integration, while the y-derivative has to match exactly the term F_y in the denominator. The insertion of higher derivatives $\partial^{r+s} F/\partial x^r \partial y^s$ under the period integral will not work, since total x-derivatives cannot be dropped if $r > 1$, while the F_y in the denominator cannot be matched if $s > 1$.

4. Summary and conclusions

We have given an elementary derivation of a set of algebraic relations satisfied by (a large family of) period integrals on non-singular Riemann surfaces, both hyperelliptic and non-hyperelliptic. These relations can be used to obtain the Picard–Fuchs equations satified by the periods. Our approach makes virtually no use at all of advanced mathematical techniques. The analysis is independent of the nature of the moduli with respect to which variations are taken, and as such it can be applied to a variety of physical and mathematical problems. We therefore hope these observations may be of interest in both physics and mathematics.

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